Unit-2nd

Buddha Series

Laplace Transform

If $F(t)$ is defined for all $t \geq 0$, then the Laplace transform of $F(t)$, denoted by $\mathcal{L}[F(t)]$ or $L[F(t)]$ or $f(s)$ and defined by

$$\mathcal{L}[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = f(s)$$

provided that, the integral exists, s is a parameter may be real or complex

Laplace transform of Some Elementary functions:

① $\mathcal{L}[1] = \frac{1}{s}$ ② $\mathcal{L}[t^n] = \frac{\lfloor n}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$; n is +ve integer

③ $\mathcal{L}[e^{at}] = \frac{1}{s-a}$; $s > a$ ④ $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$ ⑤ $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$

⑥ $\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}$ ⑦ $\mathcal{L}[\cosh at] = \frac{s}{s^2-a^2}$

Properties of Laplace transformation:

1. If $\mathcal{L}[F(t)] = f(s)$, then $\mathcal{L}[e^{at}F(t)] = f(s-a) = \left\{\mathcal{L}(F(t))\right\}_{s=(s-a)}$

2. If $\mathcal{L}[F(t)] = f(s)$, and $G(t) = \begin{cases} F(t-a) ; & t > a \\ 0 ; & t < a \end{cases}$; then $\mathcal{L}[G(t)] = e^{-as} f(s)$

3. If $\mathcal{L}(F(t)] = f(s)$, then $\mathcal{L}[F(at)] = \frac{1}{a} f(s/a) = \frac{1}{a}\left\{\mathcal{L}[F(t)]\right\}_{s=s/a}$

4. If $\mathcal{L}[F(t)] = f(s)$ and $\frac{d}{dt}F(t) = F'(t)$ then $\mathcal{L}[F'(t)] = sf(s) - F(0)$
5. If $\mathcal{L}[F(t)] = f(s)$ and $\frac{d^2}{dt^2}F(t) = F''(t)$; then $\mathcal{L}[F''(t)] = s^2 f(s) - sF(0) - F'(0)$
6. If $\mathcal{L}[F(t)] = f(s)$ and $\frac{d^n}{dt^n}F(t) = F^{(n)}(t)$ then
   $\mathcal{L}[F^{(n)}(t)] = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots\cdots F^{(n-1)}(0).$
7. If $\mathcal{L}(F(t)] = f(s)$; then $\mathcal{L}\left[\int_0^t F(t)\,dt\right] = \frac{1}{s} f(s) = \frac{1}{s}\mathcal{L}(F(t)]$
8. If $\mathcal{L}[F(t)] = f(s)$; then $\mathcal{L}[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$; $n = 1, 2, 3 \cdots\cdots$
9. If $\mathcal{L}[F(t)] = f(s)$; then, $\mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\,ds$ (if integral exists)

Q 1 Find the Laplace transform of following functions:

(i) $\frac{1-\cos t}{t}$ (AKTU 2015) (ii) $\frac{\cos at - \cos bt}{t}$ [AKTU 2017]

(iii) $t e^{-t} \cosh t$ (AKTU 2014) (iv) $\frac{e^{-t}\sin t}{t}$ [AKTU 2012]

Solutions (i) $\mathcal{L}\left[\frac{1-\cos t}{t}\right]$

Let $F(t) = 1-\cos t \Rightarrow f(s) = \mathcal{L}(F(t)) = \mathcal{L}[1] - \mathcal{L}(\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$

$$\therefore \mathcal{L}\left[\frac{1-\cos t}{t}\right] = \mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\,ds = \int_s^\infty \left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} ds$$

$$= \int_s^\infty \left\{\frac{1}{s} - \frac{1}{2}\cdot\frac{2s}{s^2+1}\right\} ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty = \lim_{s\to\infty} \log\left(\frac{s}{\sqrt{s^2+1}}\right) - \log\left(\frac{s}{\sqrt{s^2+1}}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{1}{\sqrt{1+1/s^2}}\right) + \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$= 0 + \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

Hence $\mathcal{L}\left[\frac{1-\cos t}{t}\right] = \log\left(\frac{\sqrt{s^2+1}}{s}\right) = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$ Ans

(ii) $\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \mathcal{L}\left[\frac{F(t)}{t}\right]$ where $F(t) = \cos at - \cos bt$

$$\therefore f(s) = \mathcal{L}[F(t)] = \mathcal{L}[\cos at] - \mathcal{L}[\cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\,ds = \int_s^\infty \left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\} ds$$

$$= \frac{1}{2}\int_s^\infty \left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\} ds = \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty = \frac{1}{2}\lim_{s\to\infty}\log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= 0 + \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$ Ans

(iii) $\mathcal{L}[t\,e^{-t}\cosh t] = \mathcal{L}[t\,e^{-t}\cosh t]$

$= \mathcal{L}[t\cdot e^{-t}(\frac{e^{t}+e^{-t}}{2})]$ $\cancel{= \frac{1}{2}\mathcal{L}[t e^{2t} + t]}$

$= \mathcal{L}[t(\frac{1+e^{-2t}}{2})] = \frac{1}{2}\mathcal{L}[t + t e^{-2t}]$

$= \frac{1}{2}\{\mathcal{L}[t] + \mathcal{L}[t e^{-2t}]\} = \frac{1}{2}\{\frac{1}{s^2} - \frac{d}{ds}\mathcal{L}[e^{-2t}]\}$

$= \frac{1}{2}\{\frac{1}{s^2} - \frac{d}{ds}\frac{1}{s+2}\} = \frac{1}{2}\{\frac{1}{s^2} + \frac{1}{(s+2)^2}\}$

(iv) $\mathcal{L}[\frac{e^{-t}\sin t}{t}]$ Let $F(t) = e^{-t}\sin t$ then $f(s) = \mathcal{L}[F(t)] = \mathcal{L}[e^{-t}\sin t]$

$\therefore f(s) = \mathcal{L}[e^{-t}\sin t] = \{\mathcal{L}[\sin t]\}_{s=(s+1)} = \{\frac{1}{s^2+1}\}_{s=s+1} = \frac{1}{(s+1)^2+1}$

Now

$$\mathcal{L}\{\frac{e^{-t}\sin t}{t}\} = \mathcal{L}\{\frac{F(t)}{t}\} = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$$

$$= [\tan^{-1}(s+1)]_s^\infty = \tan^{-1}\infty - \tan^{-1}(s+1)$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$ Ans

Q 2. Evaluate the following integral by using Laplace transform

(i) $\int_0^\infty t^3 e^{-t}\sin t\,dt$ [AKTU 2023] (ii) $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt$ [AKTU 2010, 2011]

(iii) $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$ [AKTU 2011] (iv) $\int_0^\infty \frac{e^{-3t}\sin t}{t}$ [AKTU 2015]

Solutions: (1) To evaluate $\int_0^\infty t^3 e^{-t}\sin t\,dt$ we consider

$F(t) = t^3\sin t$, then $f(s) = \mathcal{L}[F(t)] = \mathcal{L}[t^3\sin t] = (-1)^3\frac{d^3}{ds}\mathcal{L}[\sin t]$

$\mathcal{L}[t^3\sin t] = \mathcal{L}[F(t)] = (-1)\frac{d^2}{ds^2}\{\frac{d}{ds}\cdot\frac{1}{1+s^2}\} = (-1)(-1)\frac{d^2}{ds^2}\{\frac{2s}{(s^2+1)^2}\}$

$\mathcal{L}[t^3\sin t] = 2\frac{d}{ds}\{\frac{d}{ds}\frac{s}{(s^2+1)^2}\} = 2\frac{d}{ds}\{\frac{(s^2+1)^2\cdot 1 - s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\}$

$\mathcal{L}[t^3\sin t] = 2\frac{d}{ds}\{\frac{s^2+1-4s^2}{(s^2+1)^3}\} = 2\frac{d}{ds}\{\frac{1-3s^2}{(s^2+1)^3}\} = 2\{\frac{(s^2+1)^3\cdot(-6s) - (1-3s^2)3(s^2+1)^2 2}{(s^2+1)^6}$

$$\mathcal{L}[t^3 \sin t] = 2\left\{\frac{(s^2+1)\cdot(-6s) - (1-3s^2)\cdot 6s}{(s^2+1)^4}\right\} = 12s\left\{\frac{-s^2-1-1+3s^2}{(s^2+1)^4}\right\}$$

$$\mathcal{L}[t^3 \sin t] = \frac{12s\,(2s^2-2)}{(s^2+1)^4} = \frac{24s\,(s^2-1)}{(s^2+1)^4}$$

$$\therefore \int_0^\infty e^{-st}\, t^3 \sin t \, dt = \frac{24s(s^2-1)}{(s^2+1)^4}$$

Putting $s=1$ in the above we get

$$\int_0^\infty e^{-t}\, t^3 \sin t \, dt = \frac{24\times 1\,(1-1)}{(1+1)^4} = 0 \qquad \underline{\text{Ans}}$$

(ii) To evaluate $\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt$, we consider $F(t) = \frac{\sin^2 t}{t}$

$$\text{or } F(t) = \frac{1}{2t}(2\sin^2 t) = \frac{1}{2t}(1-\cos 2t) = \frac{1}{2}\left\{\frac{1-\cos 2t}{t}\right\}$$

$$\mathcal{L}[F(t)] = \frac{1}{2}\mathcal{L}\left[\frac{1-\cos 2t}{t}\right] = \frac{1}{2}\int_s^\infty \mathcal{L}[1-\cos 2t]\, ds = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right] = \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+4}}\right]$$

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \frac{1}{4}\log\left\{\frac{s^2}{s^2+4}\right\}$$

$$\therefore \int_0^\infty e^{-st} \frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log\left\{\frac{s^2}{s^2+4}\right\}$$

Putting $s=1$ in the above we get

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log\left(\frac{1}{5}\right) \qquad \underline{\text{Ans}}$$

(iii) To evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\, dt$, we consider $F(t) = e^{-at} - e^{-bt}$

$$\text{then } \int_0^\infty \frac{F(t)}{t}\, dt = \int_0^\infty \mathcal{L}[F(t)]\, ds = \int_0^\infty \mathcal{L}[e^{-at}-e^{-bt}]\, ds = \int_0^\infty \left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} ds$$

$$\int_0^\infty \frac{F(t)}{t}\, dt = \Big[\log(s+a) - \log(s+b)\Big]_0^\infty = \lim_{s\to\infty} \log\left(\frac{s+a}{s+b}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\, dt = \lim_{s\to\infty} \log\left\{\frac{1+a/s}{1+b/s}\right\} + \left\{\log\left(\frac{s+b}{s+a}\right)\right\} = 0 + \log\left(\frac{s+b}{s+a}\right) \qquad \underline{\text{Ans}}$$

(IV) To evaluate $\int_0^\infty e^{-3t}\frac{\sin t}{t}\,dt$ we consider $F(t)=\frac{\sin t}{t}$

Then $\mathcal{L}[F(t)] = f(s) = \mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \mathcal{L}[\sin t]\,ds$ $\left\{ \text{using } \mathcal{L}\left[\frac{F(t)}{t}\right] = \int_s^\infty f(s)\,ds \right.$

$$\mathcal{L}[F(t)] = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1} s$$

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \pi/2 - \tan^{-1} s = \cot^{-1}(s)$$

$$\therefore \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \cot^{-1}(s)$$

Putting $s=3$ we get

$$\int_0^\infty e^{-3t}\frac{\sin t}{t}\,dt = \cot^{-1}(3) \qquad \underline{\text{Ans}}$$

Laplace transform of Periodic Function

Periodic Function: A function $F(t)$ is said to be periodic function with period T if and only if $F(t+T)=F(t)\ \forall\ t$

For example; $F(t)=\sin t$, is a periodic function with period 2π

Theorem: If $F(t)$ is a periodic function with period T, then

$$\mathcal{L}[F(t)] = \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$$

Proof: Given that $F(t)$ is periodic function therefore

$$(1) \text{ ——— } F(t) = F(t+T) = F(t+2T) = F(t+3t] = \cdots \quad \forall\ t$$

Now

$$\mathcal{L}[F(t)] = \int_0^\infty e^{-st}F(t)\,dt = \int_0^T e^{-st}F(t)\,dt + \int_T^{2T} e^{-st}F(t)\,dt + \int_{2T}^{3T} e^{-st}F(t)\,dt + \cdots$$

$$\mathcal{L}[F(t)] = I + I_1 + I_2 + I_3 + \cdots \text{ ——— } (2)$$

where

$$I = \int_0^T e^{-st}F(t)\,dt;\quad I_1 = \int_T^{2T} e^{-st}F(t)\,dt;\quad I_2 = \int_{2T}^{3T} e^{-st}F(t)\,dt;\quad I_3 = \int_{3T}^{4T} e^{-st}F(t)\,dt$$

and so on.

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Solving $I_2, I_3, I_4 \ldots\ldots$

$$I_2 = \int_T^{2T} e^{-st} F(t)\, dt \quad \left\{ \begin{array}{l} \text{Let } t = u+T \text{ and } dt = du \\ \text{limit when } t = T \text{ Then } u=0 \text{ and} \\ \quad\quad \text{when } t = 2T \text{ Then } u = T \end{array} \right.$$

$$I_2 = \int_{u=0}^{u=T} e^{-s(u+T)} F(u+T)\, du \quad \left\{ \begin{array}{l} \text{by equation (1)} \\ F(u+T) = F(u) \end{array} \right.$$

$$= \int_0^T e^{-su} \cdot e^{-sT} F(u)\, du = \int_0^T e^{-sT} e^{-su} du = e^{-sT} \int_0^T e^{-su} F(u)\, du$$

$$I_2 = e^{-sT} \int_0^T e^{-su} F(u) du = e^{-sT} \int_0^T e^{-st} F(t)\, dt = e^{-sT} \cdot I$$

$$I_2 = e^{-sT} \cdot I \quad \text{——} \ (3)$$

Also

$$I_3 = \int_{2T}^{3T} e^{-st} F(t)\, dt \quad \left\{ \begin{array}{l} \text{Let } t = u+2T \Rightarrow dt = du \\ \text{limit when } t = 2T \text{ Then } u = 0 \\ \quad\quad \text{when } t = 3T \text{ Then } u = T \end{array} \right.$$

$$I_3 = \int_{u=0}^{u=T} e^{-s(u+2T)} F(u+2T)\, du \quad \left\{ \begin{array}{l} \text{by eq}^n \ (1) \\ F(u+2T) = F(u) \end{array} \right.$$

$$I_3 = e^{-2sT} \int_0^T e^{-su} F(u)\, du = e^{-2sT} \int_0^T e^{-su} F(u)\, du = e^{-2sT} \cdot I$$

$$I_3 = e^{-2sT} \cdot I \quad \text{——} \ (4)$$

Similarly $I_4 = e^{-3sT} I$, $I_5 = e^{-4sT} \cdot I$, ------

Putting these values from (3), (4), in equation (2) we get

$$\mathcal{L}[F(t)] = I + e^{-sT} I + e^{-2sT} I + e^{-3sT} I + \ldots\ldots$$

$$= I \left\{ 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + e^{-4sT} + \ldots \right\}$$

$$= I \left\{ \frac{1}{1 - e^{-sT}} \right\} \quad \left\{ \begin{array}{l} \therefore \text{ In GP} \\ S_\infty = \frac{a}{1-r} \end{array} \right.$$

$$\mathcal{L}[F(t)] = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}$$

<u>Proved</u>

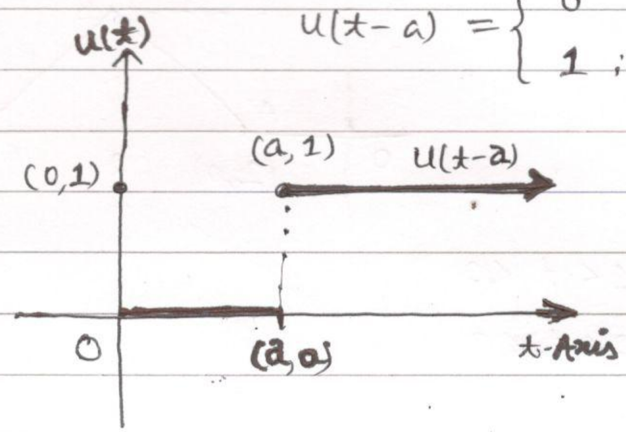
Unit step function:

The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0 & ; \ t < 0 \\ 1 & ; \ t \geq 0 \end{cases}$$

In general, if a is a +ve number, then $u(t-a)$ is the unit step function defined by

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & ; \text{if } t > a \end{cases}$$



Laplace transformation of Unit step Function

Theorem ① If $\mathcal{L}[F(t)] = f(s)$; then $\mathcal{L}[u(t-a)] = \dfrac{e^{-as}}{s}$

Proof $\mathcal{L}[F(t)] = \int_0^\infty e^{-st} F(t)\, dt = f(s)$ therefore

$$\mathcal{L}[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$$

Since $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \left[0 + \frac{e^{-sa}}{s}\right] = \frac{e^{-as}}{s}$$

Proved

Theorem ② If $\mathcal{L}[F(t)] = f(s)$, then $\mathcal{L}[F(t-a)\, u(t-a)] = e^{-as} f(s)$

Proof: $\mathcal{L}[F(t-a)\, u(t-a)] = \int_0^\infty e^{-st} F(t-a)\, u(t-a)\, dt$

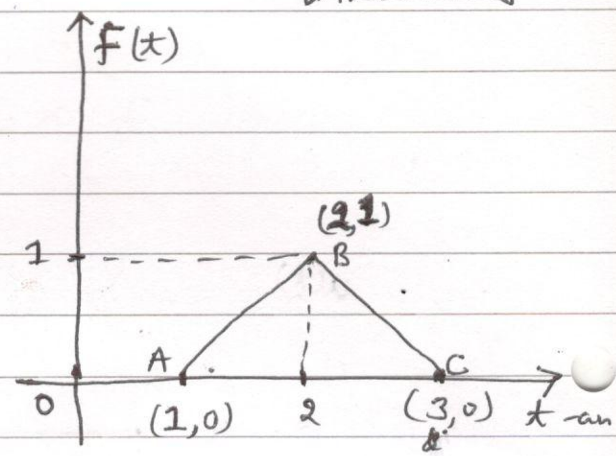
Since $F(t-a)\,u(t-a) = \begin{cases} 0 & ; t < a \\ F(t-a) & ; t > a \end{cases}$

$$= \int_0^a e^{-st} F(t-a)\, u(t-a)\, dt + \int_a^\infty e^{-st} F(t-a)\, u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)\, dt = \int_{u=0}^{u=\infty} e^{-s(a+u)} F(u)\, du = e^{-as} \int_0^\infty e^{-su} F(u)\, du = e^{-as} f(s)$$

Proved

Q3 Express the function shown in the diagram in terms of unit step function and obtain its Laplace transform [AKTU 2015]



Solution:

Equation of line AB is

$$(F(t)-0) = \frac{1-0}{2-1}(t-1)$$

$$F(t) = (t-1)$$

Equation of line BC is

$$\{F(t)-0\} = \frac{1-0}{2-3}(t-3)$$

$$F(t) = -(t-3) = 3-t$$

Thus $F(t) = \begin{cases} t-1 ; & 1<t<2 \\ 3-t ; & 2<t<3 \end{cases}$

Now F(t) can be written ~~as~~ in terms of unit step function as

$$F(t) = (t-1)\{u(t-1)-u(t-2)\} + (3-t)\{u(t-2)-u(t-3)\}$$

$$F(t) = (t-1)u(t-1) + (t-3)u(t-3) - 2(t-2)u(t-2)$$

Hence

$$\mathcal{L}[F(t)] = \mathcal{L}[(t-1)u(t-1)] + \mathcal{L}[(t-3)u(t-3)] - 2\mathcal{L}[(t-2)u(t-2)]$$

$$= \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} - \frac{2e^{-2s}}{s^2} = \frac{e^{-s}}{s^2}\{1+e^{-2s}-2e^{-s}\}$$

$$\mathcal{L}[F(t)] = \frac{e^{-s}(1-e^{-s})^2}{s^2} \quad \underline{\text{Ans}}$$

Q4 Find the Laplace transform of "saw-tooth wave" function F(t) which is periodic with period 1 and defined as $F(t) = kt$; in $0<t<1$. [AKTU 2017]

Solution we know that $\mathcal{L}[F(t)] = \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$ where F(t) is periodic with period T. Here $F(t) = kt$, and $T = 1$ we get

$$\mathcal{L}[F(t)] = \frac{1}{1-e^{-s}}\int_0^1 e^{-st}\,tK\,dt = \frac{K}{1-e^{-s}}\int_0^1 t\,e^{-st}\,dt = \frac{K}{1-e^{-s}}\left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^1$$

$$= \frac{K}{(1-e^{-s})}\left[\frac{-se^{-s}-e^{-s}}{s^2} + \frac{1}{s^2}\right] = \frac{K}{(1-e^{-s})}\left[\frac{(1-e^{-s})}{s^2} - \frac{se^{-s}}{s^2}\right]$$

$$\mathcal{L}[F(t)] = \frac{K}{s^2} - \frac{Ke^{-s}}{s(1-e^{-s})} \quad \underline{\text{Ans}}$$

Q5 Draw the graph and find the Laplace transform of the following function of period $2a$;

$$F(t) = \begin{cases} \frac{h}{a}t & ; 0 < t < a \\ \frac{h}{a}(2a-t) & ; a < t < 2a \end{cases}$$

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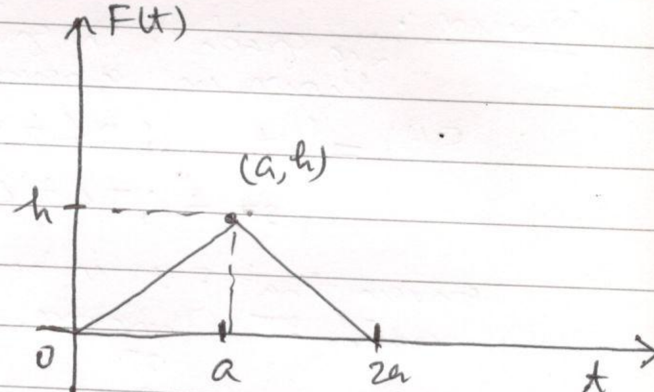
Solution:

The graph of given function

$$F(t) = \begin{cases} \frac{h}{a}t & ; 0 < t < a \\ \frac{h}{a}(2a-t) & ; a < t < 2a. \end{cases}$$

shown in figure.



Since $F(t)$ is periodic function with period $2a$. Therefore

$$\mathcal{L}[F(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}F(t)\,dt = \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st}\cdot\frac{h}{a}t\,dt + \int_a^{2a} e^{-st}\frac{h}{a}(2a\right.$$

$$\mathcal{L}[F(t)] = \frac{1}{1-e^{-2as}}\left\{\frac{h}{a}\int_0^a t\,e^{-st}dt + \frac{h}{a}\int_a^{2a}(2a-t)e^{-st}dt\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\frac{h}{a}\left[\frac{t\,e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a + \frac{h}{a}\left[(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\cdot\frac{h}{a}\left\{\left[\left(-\frac{a e^{-sa}}{s} - \frac{e^{-sa}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] + \left[\left(0 + \frac{e^{-2as}}{s^2}\right) - \left(-\frac{a e^{-as}}{s} + \frac{e^{-as}}{s^2}\right)\right]\right\}$$

$$= \frac{h}{as^2}\cdot\frac{1}{1-e^{-2as}}\left\{\cancel{-sae^{-sa}} - e^{-sa} + 1 + e^{-2as} + \cancel{sae^{-as}} - e^{-as}\right]$$

$$= \frac{h}{as^2}\frac{1}{1-e^{-2as}}\left\{e^{-2as} - 2e^{-sa} + 1\right\} = \frac{h}{as^2}\frac{1}{(1-e^{-as})(1+e^{-as})}(1-e^{-as})^2$$

$$= \frac{h}{as^2}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{h}{as^2}\frac{\cancel{e^{-as/2}}(e^{as/2}-e^{-as/2})}{\cancel{e^{-as/2}}(e^{as/2}+e^{-as/2})} = \frac{h}{as^2}\tanh\frac{as}{2}$$

Ans

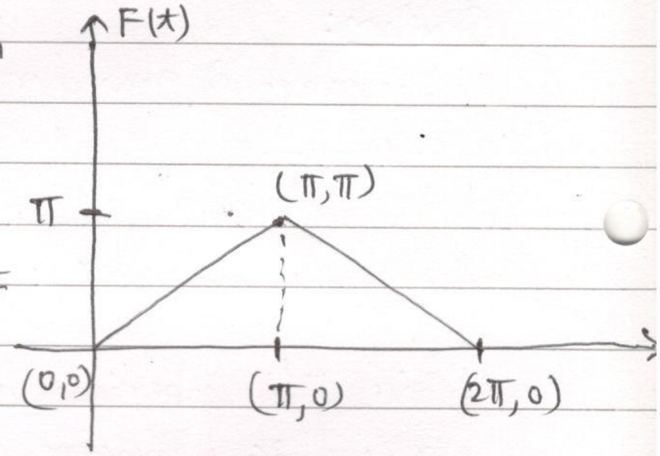
Q6. Draw the graph and find the Laplace transform of the traingular wave function of period 2π given by

$$F(t) = \begin{cases} t \;; & 0 < t < \pi \\ 2\pi - t \;; & \pi < t < 2\pi \end{cases}$$

[AKTU 2018]

<u>Solution</u>: The graph of the given function shown in figure

$$F(t) = \begin{cases} t \;; & 0 < t < \pi \\ 2\pi - t \;; & \pi < t < 2\pi \end{cases}$$



Given that $F(t)$ is periodic with period 2π Therefore

$$\mathcal{L}[F(t)] = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\, dt$$

$$\mathcal{L}[F(t)] = \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} (2\pi - t) e^{-st}\, dt \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\pi} + \left[(2\pi - t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left[\frac{-\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} \right] + \left[\left(0 + \frac{e^{-2\pi s}}{s^2}\right) - \left(\frac{-\pi e^{-\pi s}}{s} + \frac{e^{-\pi s}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{s^2} \cdot \frac{1}{1-e^{-2\pi s}} \left\{ \cancel{-s\pi e^{-s\pi}} - e^{-s\pi} + 1 + e^{-2\pi s} + \cancel{s\pi e^{-\pi s}} - e^{-\pi s} \right\}$$

$$= \frac{1}{s^2} \frac{1}{(1-e^{-s\pi})(1+e^{-\pi s})} \left(e^{-2\pi s} + 1 - 2e^{-\pi s} \right) = \frac{1}{s^2} \frac{(1-e^{-\pi s})^{\cancel{2}}}{\cancel{(1-e^{-\pi s})}(1+e^{-\pi s})}$$

$$= \frac{1}{s^2} \frac{(1-e^{-\pi s})}{(1+e^{-\pi s})} = \frac{1}{s^2} \frac{\cancel{e^{-\pi s/2}} \left(e^{\pi s/2} - e^{-\pi s/2} \right)}{\cancel{e^{-\pi s/2}} \left(e^{\pi s/2} + e^{-\pi s/2} \right)}$$

$$= \frac{1}{s^2} \tanh\left(\frac{\pi s}{2}\right)$$

<u>Ans</u>